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More Cancellation Theorems for Conjectures of Alperin and Dade

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INTRODUCTION

This paper continues and develops some of the themes of [9]. Alperin's weight conjecture (AWC), stated in [1], predicts that if G is a finite group, p is a prime, and B is a p-block of G, then we should have

$$l(B) = \sum_{Q/G} f_0^{(B)}(N_G(Q)/Q),$$

where Q/G runs through a set of representatives for the conjugacy classes of p-subgroups of G, and $f_0^{(B)}(N_G(Q)/Q)$ denotes the number projective simple $FN_G(Q)/Q$-modules in Brauer correspondents of B (when viewed as $N_G(Q)$-modules). As usual, F denotes the algebraic closure of $GF(p)$ here.

In [5], this was shown to be equivalent (for a given prime p) to the vanishing of an alternating sum formula for "ordinary" irreducible characters in a block (in the sense that one holds for all finite groups if and only if the other does). More precisely, it was shown that AWC is equivalent to

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the assertion that whenever B is a p-block of positive defect, we have

$$\sum_{\sigma \in \mathcal{N}(G,1)/G} (-1)^{|\sigma|} k(B_\sigma) = 0.$$

The notation of this paper is as in [7–9], but we briefly recap it here. $\mathcal{N}(G, 1)/G$ denotes a set of representatives for the G-orbits of (strictly increasing) normal chains of p-subgroups of G of the form $\sigma = (1 = Q_0 < \cdots Q_n)$, where $Q_i \lhd Q_n$ for all i. The action of G is the obvious conjugation action. The chain stabilizer, G_σ, of σ is the intersection of the normalizers of the p-subgroups in the chain. The sum of blocks B_σ is the sum of the blocks of G_σ in Brauer correspondence with B. As usual, $k(B) = \dim_F(Z(B))$, which is the number of complex irreducible characters associated to B, and this notation extends to a sum of blocks in a natural manner. The length of the chain σ, denoted here by $|\sigma|$, is the number of non-trivial subgroups appearing in σ.

Dade refined this Knörr–Robinson formulation of AWC in a number of ways, motivated partly by Clifford-theoretic considerations. The reformulation referred to in the title, Dade's projective conjecture (DPC), is equivalent to the following prediction:

Let G, p be as above. Let Z be a central p-subgroup of G, and let λ be a linear character of Z. Suppose that B is a p-block of G whose defect group strictly contains Z. For each non-negative integer d, let $k_d(B, \lambda)$ denote the number of complex irreducible characters, χ, assigned to B whose restriction to Z is a multiple of λ, and which satisfy $p^d \chi(1) = |G|_p$. Let $\mathcal{N}(G, Z)/G$ denotes a set of representatives for the G-orbits of (strictly increasing) normal chains of p-subgroups of G of the form $\sigma = (Z = Q_0 < \cdots < Q_n)$, where $Q_i \lhd Q_n$ for all i. Then we should have

$$\sum_{\sigma \in \mathcal{N}(G,Z)/G} (-1)^{|\sigma|} k_d(B_\sigma,\ \lambda) = 0$$

for any such choice of d and λ.

In fact, it is no real loss of generality to assume that $Z = O_p(G)$ (and implicitly that $O_p(G)$ is central). Also, as explained in [7], it is, in fact, the case that DPC in the form above is equivalent to requiring that we always have

$$\sum_{\sigma \in \mathcal{N}(G,Z)/G} (-1)^{|\sigma|} k_d(B_\sigma) = 0$$

for any choice of d as above.

Since DPC is a refinement of the Knörr–Robinson reformulation of AWC, it follows that AWC would be a logical consequence if DPC were proved.

When U is a p-subgroup of G, we let $\mathcal{N}(G, U)/N_G(U)$ denote a set of representatives for the $N_G(U)$-conjugacy classes of normal chains of p-subgroups of G whose initial subgroup is U. Notice that these are in natural bijection with G-orbits of normal chains of p-subgroups of G whose initial subgroup is G-conjugate to U.

Notice, then, that for G, B, d as before we may write

$$\sum_{\sigma \in \mathcal{N}(G,Z)/G} (-1)^{|\sigma|} k_d(B_\sigma) = k_d(B) - \sum_{U/G} \sum_{\sigma \in \mathcal{N}(G,U)/N_G(U)} (-1)^{|\sigma|} k_d(B_\sigma),$$

where U/G denotes a set of representatives of p-subgroups of G which strictly contain Z (up to G-conjugacy).

In this paper, we give a number of conditions which ensure that chains in $\mathcal{N}(G, U)/N_G(U)$ (U a p-subgroup of G, notation as in [7]) make zero contribution to the above alternating sum. As well as having some theoretical interest, when interpreted as making new predictions about control of character defects for groups in which DPC can be shown to hold in every section (in other words, pointing out consequences of DPC if it holds), such observations can simplify computations in particular examples when DPC is assumed to hold in every *proper* section (in other words, placing restrictions on the structure of a minimal counterexample, or at least, removing the necessity of considering many chains). We also point out (in a way to be made more precise later in the paper, but only outlined here) that it would follow from DPC that the same consequences of Broué's conjecture for numbers of irreducible characters in blocks with Abelian defect groups must still hold in p-power central extensions (where characters of non-zero defect (positive height) may occur), and for blocks covering such blocks of normal subgroups (but which need not themselves have Abelian defect group).

REVIEW OF COPRIME AUTOMORPHISMS OF p-GROUPS

Here we collect together some known facts about p-groups and their automorphisms (especially coprime ones) which will be useful to us later. Most of the results needed can be found in Gorenstein [3].

Let G be a finite group, and let H be a characteristic section of G (that is, H has the form L/K where L and K are both characteristic subgroups of G). We say that H is *coprime automorphism preserving* (henceforth abbreviated to CAP) for G if $C_A(H) = 1_A$ whenever A is a group of coprime automorphisms of G. Notice that if the section H is CAP for G and the section K is CAP for H, then the section K is CAP for G.

The following types of sections are known to be CAP for G:

(i) Any characteristic subgroup H of G for which $C_G(H) = Z(H)$ (in particular, $F^*(G)$ is CAP for G , and if G is a p-group, then the Thompson subgroup $J(G)$ is CAP for G).

(ii) If G is a p-group, then $\Omega(G)$ is CAP for G, where $\Omega(G) = \Omega_1(G)$ if p is odd or if G is Abelian, and $\Omega(G) = \Omega_2(G)$ if $p = 2$ and G is non-Abelian (as usual, $\Omega_i(G) = \langle x \in G\colon x^{p^i} = 1_G \rangle$).

(iii) If G is nilpotent, then $G/\Phi(G)$ is CAP for G.

Somewhat more subtle CAP subgroups of p-groups were discovered and exploited by J. G. Thompson. If G is a p-group, then G has a CAP subgroup of class at most 2 (and exponent p if p is odd or G is Abelian, of exponent 4 if $p = 2$ and G is non-Abelian) (actually, Thompson imposed slightly different conditions). There are several ways to construct these. One way which is computationally efficient is to define CAP subgroups G_n of G as follows:

$G_0 = G$; if $\Phi(G_n) \leq Z(G_n)$, set $G_\infty = \Omega(G_n)$. Otherwise, set $G_{n+1} = \Phi(G_n) C_{G_n}(\Phi(G_n))$.

Notice that G_{n+1} is a proper characteristic subgroup of G_n which contains its centralizer in G_n in the case that $\Phi(G_n) \not\leq Z(G_n)$, so that, in that case, G_{n+1} is CAP in G_n. On the other hand, if $\Phi(G_n) \leq Z(G_n)$, then G_n has class at most 2, and G_n' has exponent at most p. Hence the procedure above returns a CAP subgroup G_∞ of G of class at most 2 and exponent p or 4. Furthermore, $G_\infty/\Phi(G_\infty)$ is still a CAP section of G.

If desired, the procedure can be refined as follows (but the resulting procedure is not as computationally efficient): suppose that we have reached the point that $\Phi(G_n) \leq Z(G_n)$. If G_n has a non-central characteristic Abelian subgroup, say K_n, we set $G_{n+1} = C_{G_n}(K_n)$, which is still a CAP section of G. Hence, iterating the procedure, we can produce a CAP subgroup G_∞ which has exponent p or 4, has all characteristic Abelian subgroups central, and has central Frattini subgroup.

If a finite group G is contained in another finite group H and $S = L/K$ is a section of G, then $C_H(S)$ is defined to be $\{h \in H : [L, h] \leq K\}$ (which is clearly a normal subgroup of $N_H(K) \cap N_H(L)$).

DEFECT GROUPS, B-PARABOLICS, AND CAP SECTIONS

LEMMA. *Let U be a subgroup of the finite p-group P, and let S be a CAP section of U. Then if $C_P(S) \not\leq U$, we have $C_{N_P(U)}(S) \not\leq U$.*

Proof. Let $C = C_P(S)$. Then U normalizes C, so that UC is a subgroup of P. If $C \not\leq U$, then $UC > U$, so that $UN_C(U) = N_{UC}(U) > U$. Hence $N_C(U) \not\leq U$.

Now let B be a block of RG for a finite group G, and let D be a defect group for B. Let (D, b^*) be a maximal B-subpair. Let U be a subgroup of D, and let $(U, b) \leq (D, b^*)$ be a B-subpair such that $(N_D(U), b_1) \leq (D, b^*)$ is a Sylow subpair of $N_G(U)$, that is, a maximal B'-subpair for some Brauer correspondent of B for $N_G(U)$. Suppose further that $C_D(U) = Z(U)$.

Let S be a CAP section of U. Then

$$(C_G(S) \cap N_G(U))/C_G(U) \leq O_p(N_G(U)/C_G(U)).$$

If $C_D(S) \not\leq U$, then $C_D(S) \cap N_G(U) > U$. Now B' covers a block of $C_G(S) \cap N_G(U)$ with defect group $C_D(S) \cap N_G(U) \not\leq U$. Since B' covers a nilpotent block of $C_G(U)$ (with defect group Z(U)), we conclude that B' covers a nilpotent block of $U(C_G(S) \cap N_G(U))$ whose defect group strictly contains U.

By Theorem 3.2.2 of [9], we deduce that

$$\sum_{\sigma \in \mathcal{N}(G,U)/N_G(U)} (-1)^{|\sigma|} k_d(B'_\sigma) = 0$$

for each non-negative integer d (with notation as explained in the Introduction). We first need to note that the alternating sum vanishes by the usual elementary argument if $U \neq O_p(N_G(U))$. Similarly, the assumption that $C_D(U) \leq U$ above is no real loss of generality, since it is proved in [9] that, in the situations to be considered below, the alternating sum in question vanishes anyway if this is not the case.

Hence we have:

THEOREM A. *Let G be a minimal counterexample to DPC* (*in the sense described in* [7]), *and let B be a block of RG with defect group D for which some formula predicted by DPC fails. Let Z be as in the Introduction, and let $U > Z$ be a p-subgroup of G for which*

$$\sum_{\sigma \in \mathcal{N}(G,U)/N_G(U)} (-1)^{|\sigma|} k_d(B_\sigma) \neq 0$$

for some non-negative integer d. Then there is a conjugate, V say, of U, contained in D such that $C_D(S) \leq V$ for each CAP section, S of V.

Remark. Once more, this can be recast as a theorem about groups in which DPC is valid in every section. (In a minimal counterexample, we are really assuming that DPC holds in every *proper* section of the group in question, and deducing consequences which hold if and only if the conjecture is valid for the group itself. Putting this another way, if we knew that

DPC were valid for *every* section of the group (including the group itself), we could conclude that the consequences held.) The same remark applies to the two theorems below. In fact, the theorem above can be strengthened somewhat: for, setting $\overline{G} = G/Z$, etc., then if U is as above, and $\overline{S}$ is a CAP section of $\overline{U}$, then

$$(C_G(\overline{S}) \cap N_G(U))/C_G(U)$$

is still a p-group (for if $y \in N_G(U)$ is p-regular with $\bar{y} \in C_{\overline{G}}(\overline{S})$, then $[U, y] \leq Z$, so that $[U, y, y] = 1$).

Hence we have:

THEOREM B. *Let G be a minimal counterexample to DPC and let B be a block of RG with defect group D for which some formula predicted by DPC fails. Let $U > Z$ be a p-subgroup of G for which*

$$\sum_{\sigma \in \mathcal{N}(G,U)/N_G(U)} (-1)^{|\sigma|} k_d(B_\sigma) \neq 0$$

for some non-negative integer d. Then there is a conjugate, V say, of U, contained in D such that $C_D(\overline{S}) \leq V$ for each CAP section, $\overline{S}$, of $\overline{V}$, where $\overline{G} = G/Z$, etc.

Theorem B admits yet further strengthening. For let G be as above, and let N be a non-central normal subgroup of G, containing Z. Let B cover a block B_1 with defect group D_0 of N (we know from [9] that $D_0 > Z$). Let us consider a p-subgroup $U > Z$ of N, and the contribution to the relevant alternating sum in Theorem 1 of [9] from (orbits of) chains starting with p-subgroups V of G with $V \cap N = U$. This is the same (for a given d) as

$$\sum_{\sigma \in \mathcal{N}(N,U)/N_G(U)} (-1)^{|\sigma|} k_d(B_\sigma)$$

by the elementary arguments outlined in [9].

This last alternating sum is 0 by Theorem 2 of [9], unless we choose may choose a conjugate of U, say W, so that $C_{D_0}(W) \leq W$. Hence we may suppose that $C_{D_0}(U) = Z(U)$, and that $N_{D_0}(U)$ is a defect group for some Brauer correspondent for $RN_N(U)$ of a block of RN covered by B. By Theorem 3.2.2 of [9], for a given p-subgroup V of G with $V \cap N = U$, the contribution from chains in $\mathcal{N}(N_G(V), V)/N_G(V)$ will be zero if some Brauer correspondent of B for $N_G(V)$ covers a nilpotent block of a normal subgroup with defect group not contained in U. By the arguments above, there will be such a block unless we can choose a conjugate, W, of U so that $C_D(\overline{S}) \leq W$ for each CAP section, $\overline{S}$, of $\overline{W}$, where $\overline{N} = N/Z$, etc.

THEOREM C. *Let G be a minimal counterexample to DPC and let B be a block of RG with defect group D for which some formula predicted by DPC fails. Let N be a non-central normal subgroup of G. Let $D_0 = D \cap N$. Let $U > Z$ be a p-subgroup of N for which*

$$\sum_{\sigma \in \mathcal{N}(N,U)/N_G(U)} (-1)^{|\sigma|} k_d(B_\sigma) \neq 0$$

for some non-negative integer d. Then there is a conjugate, W say, of U, contained in D_0 such that $C_{D_0}(\overline{S}) \leq W$ for each CAP section, $\overline{S}$, of $\overline{W}$, where $\overline{N} = N/Z$, etc.

EXAMPLES, APPLICATIONS, PARTIAL RESULTS

Since Dade's projective conjecture is proved in [7] for p-solvable groups, we may deduce:

THEOREM D. *Let G be a finite p-solvable group such that $O_p(G) = Z \leq Z(G)$. Let B be a block of RG with defect group D. Let N be a non-central normal subgroup of G containing Z, and let $D_0 = D \cap N$. Let $U > Z$ be a p-subgroup of N for which*

$$\sum_{\sigma \in \mathcal{N}(N,U)/N_G(U)} (-1)^{|\sigma|} k_d(B_\sigma) \neq 0$$

for some non-negative integer d. Then there is a conjugate, W say, of U, contained in D_0 such that $C_{D_0}(\overline{S}) \leq W$ for each CAP section, $\overline{S}$, of $\overline{W}$, where $\overline{N} = N/Z$, etc.

Perhaps worth singling out is:

COROLLARY E. *Let G be a minimal counterexample to DPC and let B be a block of RG with defect group for which some formula predicted by DPC fails. Let $U > Z$ be a p-subgroup of G for which*

$$\sum_{\sigma \in \mathcal{N}(G,U)/N_G(U)} (-1)^{|\sigma|} k_d(B_\sigma) \neq 0$$

for some non-negative integer d. Then:

(i) *U/Z is not cyclic.*

(ii) *If $\Phi(U/Z)$ contains every elementary Abelian normal subgroup of U/Z, then U/Z contains every elementary Abelian subgroup of D/Z which it normalizes.*

Proof. Set $\overline{G} = G/Z$, etc. If $\overline{U}$ is cyclic, then $N_{\overline{D}}(\overline{U}) \leq C_{\overline{D}}(\overline{U}/\Phi(\overline{U}))$. By Theorem B, we must have $\overline{D} = \overline{U}$. However, in that case, Dade has proved in [2] that G is not a counterexample to DPC. If $\Phi(\overline{U})$ contains every elementary Abelian normal subgroup of $\overline{U}$, then let $\overline{A}$ be any elementary Abelian subgroup of $\overline{D}$ normalized by $\overline{U}$. Suppose that $\overline{A} \not\leq \overline{U}$. Since $\overline{U}\,\overline{A}$ is a p-group strictly containing $\overline{U}$, we have $\overline{X} = N_{\overline{A}}(\overline{U}) \not\leq \overline{U}$. Then $[\overline{U}, \overline{X}] \leq \overline{U} \cap \overline{X} \leq \Phi(\overline{U})$ by hypothesis (as $[\overline{U}, \overline{X}]$ is an elementary Abelian normal subgroup of $\overline{U}$). However, by Theorem B, we have $C_{\overline{D}}(\overline{U}/\Phi(\overline{U})) \leq \overline{U}$, a contradiction.

Remarks. The analogue of part (i) of Corollary E was already observed for Alperin's conjecture in [5]. Notice that (using P. Hall's classification of p-groups of symplectic type) we may conclude from Corollary E that if $\overline{U}$ is (generalized) quaternion, then $\overline{D}$ must be semi-dihedral or generalized quaternion (and if $\overline{U}$ is not quaternion of order 8 with 3 dividing $[N_{\overline{G}}(\overline{U}) : C_{\overline{G}}(\overline{U})]$, then $\overline{U} = \overline{D}$).

Similar statements to Corollary E may be made about defect groups of covered blocks, but we do not state them explicitly.

We may turn Theorem C around as follows:

THEOREM F. *Let G be a minimal counterexample to DPC and let B be a block of RG with defect group D for which some formula predicted by DPC fails. Let N be a non-central normal subgroup of G containing Z and let $D_0 = D \cap N$. Suppose that there is no proper subgroup U of D_0 such that $C_{D_0}(\overline{S}) \leq U$ for each CAP section, $\overline{S}$, of $\overline{U}$ (where $\overline{N} = N/Z$, etc). Then there must be some non-negative integer d such that $k_d(B) \neq k_d(B')$, where B' is the Harris–Knörr correspondent* (*see* [4]) *of B for $N_G(D_0)$.*

Remarks. Even when D/Z is Abelian and $N = G$, Theorem F says something interesting. For then it may be re-cast as saying that in that situation, DPC predicts that there should be a defect preserving bijection between irreducible characters in B and irreducible characters in the unique Brauer correspondent for $N_G(D)$. When $Z = 1$, this is a predicted consequence of Broué's conjecture for blocks with Abelian defect groups.

It follows from the results of [8] that if G is a minimal counterexample to DPC, but Z is non-trivial, and D/Z is Abelian, then there is a bijection between irreducible characters in B and irreducible characters in the unique Brauer correspondent of B for $N_G(D)$, but it is not clear to us at present how to incorporate character defects to get the full strength of the desired refinement. Alternatively, if Broué's conjecture were proved (in the weaker perfect isometry version), then it would follow from [8] that there is a bijection between irreducible characters in B and irreducible characters in the unique Brauer correspondent for $N_G(D)$ in the case that $D/O_p(Z(G))$ is Abelian.

We remark without proof that, by arguments similar to the above, we have:

THEOREM G. *Let G be a minimal counterexample to DPC and let B be a block of RG with defect group D for which some formula predicted by DPC fails. Let N be a non-central normal subgroup of G containing Z, and let $D_0 = D \cap N$. Let $\overline{B}$ be the unique block of $\overline{G} = G/Z$ which corresponds to B under the canonical epimorphism from RG onto $R\overline{G}$. Then if $\overline{B}$ covers blocks of $\overline{N}$ which have TI defect groups, there must be some non-negative integer d such that $k_d(B) \neq k_d(B')$, where B' is the Harris–Knörr correspondent of B for $N_G(D_0)$.*

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